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THE BENJAMIN-BONA-MAHONY EQUATION WITH DISSIPATIVE MEMORY

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Abstract. We show that the nonlinear contraction semigroup generated by the Benjamin-Bona-Mahony equation with dissipative memory

$$u_t - u_{txx} + u_x - \int_0^\infty g(s) u_{xx}(t-s)\,\mathrm{d}s + u^p u_x = 0$$

is exponentially stable for every $p \in \mathbb{N}$.

1. Introduction

This paper deals with the propagation of the one-directional small amplitude long waves in shallow water. In the conservative context, such waves are described by the Korteweg-de Vries (KdV) equation [16]

$$u_t + u_{xxx} + u_x + uu_x = 0,$$

where $u = u(x,t) : I \times \mathbb{R}^+ \to \mathbb{R}$ denotes the wave surface, $I \subset \mathbb{R}$ being a bounded interval. In 1972, Benjamin, Bona and Mahony [4] proposed to replace the term u_{xxx} by $-u_{txx}$, thus obtaining the regularized KdV equation (here called BBM equation)

$$u_t - u_{txx} + u_x + uu_x = 0.$$

The equation above can be directly derived from Newton's second law, in the same way the KdV equation is obtained from the Euler one [18, 19]. In the dissipative context, the BBM equation turns into

$$u_t - u_{txx} - \nu u_{xx} + u_x + uu_x = 0, \quad \nu > 0, \tag{1.1}$$

or, more generally,

$$u_t - u_{txx} - \nu u_{xx} + (f(u))_x = q \tag{1.2}$$

where f and q are a suitable nonlinear function and a time-independent forcing term, respectively. Actually, it is a standard matter to prove that the initial value problem associated to (1.2) with the Dirichlet boundary condition is globally well-posed in the Sobolev space $H_0^1(I)$. Hence, it generates a nonlinear solution semigroup $S(t)$ on $H_0^1(I)$ defined by the action

$$u_0 \mapsto S(t)u_0 = u(t),$$

where $u(t)$ is the unique solution at time t with initial datum $u_0 \in H_0^1(I)$. Concerning the longtime dynamics, Wang and Yang [26, 28] proved the existence of a finite-dimensional global attractor for $S(t)$. Since the semigroup is not compact in $H_0^1(I)$, the proof is based

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on the weak continuity of $S(t)$ and energy methods inspired by Ghidaglia's work (see e.g. [10, 24, 25]). Other results can be found for instance in [3, 5, 13, 17, 24, 27] and references therein.

Coming back to the homogeneous model (1.1), multiplying in $L^2(I)$ the equation by $2u$ and exploiting the Dirichlet boundary condition, the (twice) energy

$$E(t) = ||S(t)u_0||^2_{H^1(I)}$$

is readily seen to satisfy the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -2\nu ||u_x(t)||^2_{L^2(I)}.$$

Hence, in light of the Poincaré inequality and the Gronwall lemma, we deduce the exponential stability

$$E(t) \leq E(0)\mathrm{e}^{-\kappa t},$$

where κ is a strictly positive constant depending only on ν and the interval I. Note that, in the conservative limit case $\nu = 0$, the energy is preserved, namely $E(t) = E(0)$. Many other papers related with damped BBM equations with weaker dissipation are nowadays present in the literature (see [1, 2, 6, 15]). Still, to the best of our knowledge, none of them is dealing with dispersive equations with dissipative memory.

Motivated by the discussion above, our aim is to study the asymptotic behavior of the integro-differential equation

$$u_t - u_{txx} + u_x - \int_0^\infty g(s) u_{xx}(t-s)\,\mathrm{d}s + u^p u_x = 0 \tag{1.3}$$

in the unknown $u = u(x,t) : I \times \mathbb{R} \to \mathbb{R}$, complemented with the Dirichlet boundary condition

$$u_{|\partial I} = 0.$$

Here $p \in \mathbb{N}$ is a fixed constant (when $p = 0$ the model becomes linear), while g is a bounded convex summable function on $[0,\infty)$ of total mass

$$\int_0^\infty g(s)\,\mathrm{d}s = 1$$

having the explicit form

$$g(s) = \int_s^\infty \mu(y)\,\mathrm{d}y,$$

where $\mu : \mathbb{R}^+ = (0,\infty) \to [0,\infty)$, the so-called memory kernel, is a nonincreasing absolutely continuous summable function of total mass

$$\varkappa := \int_0^\infty \mu(s)\,\mathrm{d}s = g(0) > 0.$$

Moreover, the function u is supposed to be known for all $t \leq 0$. From the physical viewpoint, equation (1.3) can be interpreted as a memory relaxation of the dissipative BBM model (1.1) which, setting $\nu = 1$, is formally recovered when $p = 1$ and the kernel g collapses into the Dirac mass at zero. It is also worth noting that the memory term provides a more realistic description of the Fick's law. In particular, it prevents the infinite propagation speed of regularization [8, 23]. In this work we prove that the nonlinear solution semigroup generated by (1.3), acting on a suitable Hilbert space accounting for the presence of the memory, remains exponentially stable.

In order to explain the mathematical difficulties encountered in the analysis, we begin to observe that, also at a linear level, the exponential stability of (1.3) is much harder to prove than the one of (1.1). An enlightening example is provided by a comparison between the classical heat equation

$$u_t - u_{xx} = 0$$

with the Dirichlet boundary condition and its memory relaxation, i.e. the Gurtin-Pipkin equation [14]

$$u_t - \int_0^\infty g(s)u_{xx}(t-s)\,\mathrm{d}s = 0.$$

In the first case, similarly to (1.1), the exponential stability is almost trivial, whereas the exponential stability of the Gurtin-Pipkin model has been proved only in recent years [11]. In the nonlinear situation the picture is even worse. Indeed, although the asymptotic analysis of the one-dimensional reaction-diffusion equation is carried out under quite general assumptions, the corresponding nonlinear Gurtin-Pipkin case suffers from serious drawbacks, and requires the choice of specific memory kernels concentrated at zero [12]. For the BBM equation the scenario is similar: while adding a further nonlinearity $h(u)$ in (1.2) does not cause any essential extra difficulty, the picture becomes much more involved when dissipative memory is introduced. In particular, even showing exponential stability in the homogeneous case (as we do in the present paper) is not at all an easy task. Concerning the existence of the global attractor when further nonlinearities and/or source terms are present, the techniques devised in this work do not apply and, at the moment, an answer seems out of reach.

Plan of the paper. In the next Section 2 we introduce the functional setting and the notation, while in Section 3 we establish the existence of the solution semigroup. The final Sections 4 and 5 are devoted to the main result about exponential stability.

2. Functional Setting and Notation

In what follows, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ will denote the standard inner product and norm on the Hilbert space $L^2(I)$. In order to simplify the calculation, we introduce the strictly positive operators

$$A = -\partial_{xx} \qquad \text{with} \qquad \mathfrak{D}(A) = H^2(I)\cap H^1_0(I) \Subset L^2(I)$$

and

$$B = I + A \qquad \text{with} \qquad \mathfrak{D}(B) = \mathfrak{D}(A).$$

The operator B commutes with A and the bilinear form

$$\langle u,v\rangle_1 = \langle B^{\frac{1}{2}}u, B^{\frac{1}{2}}v\rangle$$

defines an equivalent inner product on the space $H^1_0(I)$ with induced norm

$$\|u\|_1^2 = \|u\|^2 + \|u_x\|^2,$$

and we have the Poincaré inequality

$$\frac{\lambda_1}{1+\lambda_1}\|u\|_1^2 \leq \|u_x\|^2 \tag{2.1}$$

where $\lambda_1 > 0$ is the first eigenvalue of A. Finally, we consider the so-called memory space

$$\mathcal{M} = L^2_\mu(\mathbb{R}^+; H^1_0(I))$$

of square summable H^1_0-valued functions on $\mathbb{R}^+$ with respect to the measure $\mu(s)\mathrm{d}s$, endowed with the inner product

$$\langle \eta, \xi \rangle_{\mathcal{M}} = \int_0^\infty \mu(s) \langle \eta_x(s), \xi_x(s) \rangle \,\mathrm{d}s.$$

The infinitesimal generator of the right-translation semigroup on $\mathcal{M}$ is the linear operator

$$T\eta = -\eta'$$

with domain

$$\mathfrak{D}(T) = \Big\{ \eta \in \mathcal{M} : \eta' \in \mathcal{M},\ \lim_{s\to 0} \|\eta_x(s)\| = 0 \Big\},$$

the *prime* standing for the weak derivative with respect to the internal variable $s \in \mathbb{R}^+$. Defining the nonnegative functional

$$\Gamma[\eta] = -\int_0^\infty \mu'(s) \|\eta_x(s)\|^2 \,\mathrm{d}s,$$

an integration by parts together with a limiting argument yield the equality (see [7, 21])

$$2\langle T\eta, \eta \rangle_{\mathcal{M}} = -\Gamma[\eta]. \tag{2.2}$$

The phase space of our problem will be

$$\mathcal{H} = H^1_0(I) \times \mathcal{M}$$

endowed with the norm

$$\|(u,\eta)\|^2_{\mathcal{H}} = \|u\|_1^2 + \|\eta\|^2_{\mathcal{M}}.$$

3. The Contraction Semigroup

We translate the problem in the so-called history space framework of Dafermos [8]. To this aim, introducing the auxiliary variable

$$\eta = \eta^t(x,s) = \int_0^s u(x, t-y)\,\mathrm{d}y,$$

accounting for the integrated past history of u, we rewrite (1.3) as

$$Bu_t + u_x + \int_0^\infty \mu(s) A\eta(s)\,\mathrm{d}s + u^p u_x = 0, \tag{3.1}$$

$$\eta_t = T\eta + u. \tag{3.2}$$

By means of standard arguments based on a Galerkin approximation procedure, one can show that system (3.1)-(3.2) above is well-posed in the phase space $\mathcal{H}$. In particular, the solution continuously depends on the initial data. As a consequence, it generates a strongly continuous solution semigroup

$$S(t) : \mathcal{H} \to \mathcal{H}$$

defined by the action

$$z_0 = (u_0, \eta_0) \mapsto S(t)z_0 = z(t),$$

where

$$z(t) = (u(t), \eta^t)$$

is the unique (weak) solution to (3.1)-(3.2) with initial datum $z(0) = z_0$. Introducing (twice) the *energy* at time $t \geq 0$ corresponding to the initial datum $z_0 \in \mathcal{H}$ as

$$E(t) = \|S(t)z_0\|^2_{\mathcal{H}},$$

we multiply (3.1) by $2u$ in $L^2(I)$ and (3.2) by 2η in $\mathcal{M}$. Summing up, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E + 2\langle u_x, u\rangle + 2\langle u^p u_x, u\rangle = 2\langle T\eta, \eta\rangle_{\mathcal{M}}.$$

Since, due to the boundary condition,

$$2\langle u_x, u\rangle + 2\langle u^p u_x, u\rangle = \int_I \frac{\mathrm{d}}{\mathrm{d}x}(u^2(x))\,\mathrm{d}x + \frac{2}{p+2}\int_I \frac{\mathrm{d}}{\mathrm{d}x}(u^{p+2}(x))\,\mathrm{d}x = 0,$$

an exploitation of (2.2) provides the *energy identity*

$$\frac{\mathrm{d}}{\mathrm{d}t} E + \Gamma[\eta] = 0. \tag{3.3}$$

In particular, since the functional $\Gamma[\eta]$ is nonnegative, we have the control

$$E(t) \leq E(0), \tag{3.4}$$

meaning that $S(t)$ is actually a *contraction semigroup.*

4. Exponential Stability

For the longterm analysis, the memory kernel μ is supposed to satisfy the additional assumption (see [8])

$$\mu'(s) + \delta\mu(s) \leq 0 \tag{4.1}$$

for some $\delta > 0$ and almost every $s \in \mathbb{R}^+$. Note that μ can be unbounded in a neighborhood of zero.

The main result of the paper reads as follows.

Theorem 4.1. *There exist a strictly positive constant ω, depending on μ and the length of the interval $|I|$, and an increasing positive function $\mathcal{Q}_p$, depending besides on μ and $|I|$ also on p, such that*

$$E(t) \leq \mathcal{Q}_p(E(0))\mathrm{e}^{-\omega t}.$$

In order to prove Theorem 4.1, we need to introduce an auxiliary energy-type functional. First, due to the possible singularity of μ at zero, we choose $s_\star > 0$ such that

$$\int_0^{s_\star} \mu(s)\,\mathrm{d}s \leq \frac{\varkappa}{4}. \tag{4.2}$$

Then, defining the truncated kernel

$$\rho(s) = \mu(s_\star)\chi_{(0,s_\star]}(s) + \mu(s)\chi_{(s_\star,\infty)}(s),$$

for $\varepsilon > 0$ we consider the functional

$$\Psi_\varepsilon(t) = -\varepsilon \int_0^\infty \rho(s)\langle u_x(t), \eta^t_x(s)\rangle\,\mathrm{d}s.$$

Since $\rho(s) \leq \mu(s)$, it is easily seen that

$$|\Psi_\varepsilon(t)| \leq \alpha\varepsilon E(t) \tag{4.3}$$

for every $t \geq 0$, for some universal constant $\alpha = \alpha(\mu, |I|) > 0$.

Lemma 4.2. *There exist universal constants $\beta, \gamma > 0$, depending only on μ and $|I|$ but independent on p and the initial energy $E(0)$, such that the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_\varepsilon(t) + \frac{\varepsilon\varkappa}{4}\|u_x(t)\|^2 \leq \frac{\delta}{4}\|\eta^t\|^2_{\mathcal{M}} + \beta\varepsilon\Gamma[\eta^t] \tag{4.4}$$

holds for every $t \geq 0$, whenever $\varepsilon E(0)^p \leq \gamma$.

Proof. In what follows $C \geq 0$ will denote a *generic* constant possibly depending on the structural quantities of the problem but independent on p and the initial energy $E(0)$. We compute the time derivative of Ψ_ε as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Psi_\varepsilon &= -\varepsilon\int_0^\infty \rho(s)\langle u_{tx}, \eta_x(s)\rangle\,\mathrm{d}s - \varepsilon\int_0^\infty \rho(s)\langle u_x, \eta_{tx}(s)\rangle\,\mathrm{d}s \\
&= \varepsilon\int_0^\infty \rho(s)\langle B^{-1}u_x, A\eta(s)\rangle\,\mathrm{d}s + \varepsilon\int_0^\infty \rho(s)\Big(\int_0^\infty \mu(\sigma)\langle B^{-1}A\eta(\sigma), A\eta(s)\rangle\,\mathrm{d}\sigma\Big)\,\mathrm{d}s \\
&\quad + \varepsilon\int_0^\infty \rho(s)\langle B^{-1}(u^p u_x), A\eta(s)\rangle\,\mathrm{d}s - \varepsilon\int_0^\infty \rho(s)\langle u_x, T\eta_x(s)\rangle\,\mathrm{d}s - \varepsilon\|u_x\|^2\int_0^\infty \rho(s)\mathrm{d}s.
\end{aligned}$$

Then appealing to (4.1) we estimate

$$\begin{aligned}
&\varepsilon\int_0^\infty \rho(s)\langle B^{-1}u_x, A\eta(s)\rangle\,\mathrm{d}s + \varepsilon\int_0^\infty \rho(s)\Big(\int_0^\infty \mu(\sigma)\langle B^{-1}A\eta(\sigma), A\eta(s)\rangle\,\mathrm{d}\sigma\Big)\,\mathrm{d}s \\
&\leq C\varepsilon\big(\|u_x\|\|\eta\|_{\mathcal{M}} + \|\eta\|^2_{\mathcal{M}}\big) \\
&\leq \frac{\varkappa\varepsilon}{8}\|u_x\|^2 + C\varepsilon\|\eta\|^2_{\mathcal{M}} \\
&\leq \frac{\varkappa\varepsilon}{8}\|u_x\|^2 + C\varepsilon\Gamma[\eta].
\end{aligned} \tag{4.5}$$

Moreover, using (4.2) and the equality $\rho(s) = \mu(s)$ for $s \geq s_\star$, we have

$$-\varepsilon\|u_x\|^2\int_0^\infty \rho(s)\,\mathrm{d}s \leq -\varepsilon\|u_x\|^2\int_{s_\star}^\infty \mu(s)\,\mathrm{d}s \leq -\frac{3\varkappa\varepsilon}{4}\|u_x\|^2. \tag{4.6}$$

Integrating by parts in s, we infer that

$$
\begin{aligned}
-\varepsilon\int_0^\infty \rho(s)\langle u_x, T\eta_x(s)\rangle\,\mathrm{d}s &= -\varepsilon\int_{s_\star}^\infty \mu'(s)\langle u_x,\eta_x(s)\rangle\,\mathrm{d}s\\
&\leq \varepsilon\|u_x\|\Big(-\int_{s_\star}^\infty \mu'(s)\|\eta_x(s)\|\,\mathrm{d}s\Big)\\
&\leq \frac{\varkappa\varepsilon}{8}\|u_x\|^2 + C\varepsilon\Big(-\int_{s_\star}^\infty \mu'(s)\|\eta_x(s)\|\,\mathrm{d}s\Big)^2.
\end{aligned}
$$

Therefore, since

$$
\Big(-\int_{s_\star}^\infty \mu'(s)\|\eta_x(s)\|\,\mathrm{d}s\Big)^2 \leq \int_{s_\star}^\infty \mu'(s)\,\mathrm{d}s\int_{s_\star}^\infty \mu'(s)\|\eta_x(s)\|^2\,\mathrm{d}s \leq \mu(s_\star)\Gamma[\eta],
$$

we obtain

$$
-\varepsilon\int_0^\infty \rho(s)\langle u_x, T\eta_x(s)\rangle\,\mathrm{d}s \leq \frac{\varkappa\varepsilon}{8}\|u_x\|^2 + C\varepsilon\Gamma[\eta]. \tag{4.7}
$$

Finally, exploiting the embedding $H^1(I)\subset L^\infty(I)$ and (3.4),

$$
\begin{aligned}
\varepsilon\int_0^\infty \rho(s)\langle B^{-1}(u^p u_x), A\eta(s)\rangle\,\mathrm{d}s &\leq C\varepsilon\|u^p u_x\|\|\eta\|_{\mathcal{M}} \leq C\varepsilon\|u\|_\infty^p\|u_x\|\|\eta\|_{\mathcal{M}}\\
&\leq C\varepsilon\|u\|_1^p\|u_x\|\|\eta\|_{\mathcal{M}} \leq C\varepsilon E(0)^{\frac{p}{2}}\|u_x\|\|\eta\|_{\mathcal{M}}\\
&\leq \frac{\delta}{4}\|\eta\|_{\mathcal{M}}^2 + C\varepsilon^2E(0)^p\|u_x\|^2.
\end{aligned}
$$

At this point, choosing $\varepsilon>0$ such that

$$
C\varepsilon E(0)^p \leq \frac{\varkappa}{4}
$$

the inequality above turns into

$$
\varepsilon\int_0^\infty \rho(s)\langle B^{-1}(u^p u_x), A\eta(s)\rangle\,\mathrm{d}s \leq \frac{\delta}{4}\|\eta\|_{\mathcal{M}}^2 + \frac{\varepsilon\varkappa}{4}\|u_x\|^2. \tag{4.8}
$$

Collecting (4.5)-(4.8), the proof is finished. □

Remark 4.3. Observe that the constants α,β,γ can be explicitly calculated in terms of the structural quantities of the problem, even in an optimal way.

We are now in a position to prove Theorem 4.1. First we consider the energy identity (3.3) which, in light of (4.1), yields

$$
\frac{\mathrm{d}}{\mathrm{d}t}E + \frac{\delta}{2}\|\eta\|_{\mathcal{M}}^2 + \frac{1}{2}\Gamma[\eta] \leq 0.
$$

Next, setting

$$
\Lambda_\varepsilon(t) = E(t) + \Psi_\varepsilon(t)
$$

and taking the sum of (4.4) with the inequality above, we obtain the estimate (valid whenever $\varepsilon E(0)^p \leq \gamma$)

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \frac{\varepsilon\varkappa}{4}\|u_x\|^2 + \frac{\delta}{4}\|\eta\|_{\mathcal{M}}^2 + \Big(\frac{1}{2}-\beta\varepsilon\Big)\Gamma[\eta] \leq 0.
$$

Due to (2.1) and (4.3), it is apparent to see that fixing[1]

$$\varepsilon = \min\left\{\frac{1}{2\alpha}, \frac{1}{2\beta}, \frac{\delta}{\varkappa}, \frac{\gamma}{E(0)^p}\right\}$$

and calling

$$\varpi = \frac{\varkappa\lambda_1}{8(1+\lambda_1)} > 0,$$

the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varpi\varepsilon\Lambda_\varepsilon \leq 0$$

holds. Hence, applying the Gronwall lemma and (4.3) once more, we infer that

$$E(t) \leq 4E(0)\mathrm{e}^{-\varpi\varepsilon t}.$$

We now set

$$t_0 = \frac{\log_+(4E(0))}{\varpi\varepsilon}.$$

Note that t_0, besides on $|I|, \mu, \alpha, \beta, \gamma$, depends also on $E(0)$ and the exponent p. However, it is clear that for every $t \geq t_0$

$$E(t) \leq 1,$$

hence, by the semigroup property,

$$E(t) = \|S(t)z_0\|^2_{\mathcal{H}} = \|S(t-t_0)S(t_0)z_0\|^2_{\mathcal{H}} \leq 4\mathrm{e}^{\omega t_0}\mathrm{e}^{-\omega t}, \quad \forall t \geq t_0$$

for some positive ω, which now is independent of p and $E(0)$. On the other hand, in light of (3.4),

$$E(t) \leq E(0)\mathrm{e}^{\omega t_0}\mathrm{e}^{-\omega t}, \quad \forall t < t_0.$$

In summary, defining

$$\mathcal{Q}_p(E(0)) = \max\{4, E(0)\}\mathrm{e}^{\omega t_0},$$

the conclusion follows. □

5. Further Remarks

I. Up to minor modifications, it is possible to allow the presence of (even infinitely many) jumps in the memory kernel μ. Indeed, denoting with $\{s_n\}_{n\geq 1}$ the increasing sequence of discontinuity points of μ and setting

$$\mu_n = \mu(s_n^-) - \mu(s_n^+) \geq 0,$$

we still have the energy identity (3.3) with

$$\Gamma[\eta] = -\int_0^\infty \mu'(s)\|\eta_x(s)\|^2\,\mathrm{d}s + \sum_n \mu_n\|\eta_x(s_n)\|^2 \geq 0.$$

In turn, the conclusions of Lemma 4.2 and Theorem 4.1 remain true (see e.g. [20]).

[1] If $E(0) = 0$ we can take any ε.

II. Condition (4.1) can be relaxed: Theorem 4.1 holds even if the kernel μ fulfills for some $C \geq 1$ and $\delta > 0$ the weaker assumption

$$\mu(t+s) \leq C\mathrm{e}^{-\delta t}\mu(s), \tag{5.1}$$

for every $t \geq 0$ and almost every $s \in \mathbb{R}^+$, provided that μ is not *too flat* (cf. [9, 20]).

III. In the linear case (i.e. when $p = 0$) exponential decay can be shown within optimal assumptions on μ, by means of linear techniques (see [22]). In this situation, besides (5.1), it is sufficient to assume that the kernel is not *completely flat*, namely, the set

$$\mathfrak{D} = \{s \in \mathbb{R}^+ : \mu'(s) < 0\}$$

has positive Lebesgue measure.

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